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The second lowest two-sided cell in an affine Weyl group ☆

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ABSTRACT

Let W_a be an irreducible affine Weyl group with W_0 the associated Weyl group. The present paper is to study the second lowest two-sided cell Ω_{qr} of W_a. Let n_{qr} be the number of left cells of W_a in Ω_{qr}. We conjecture that the equality $n_{\mathrm{qr}} = \frac{1}{2}|W_0|$ should always hold. When W_a is either $\widetilde{A}_{n-1}$, $n \geqslant 2$, or of rank $\leqslant 4$, this equality can be verified by the existing data (see 0.3). Then the main result of the paper is to prove the inequality $n_{\mathrm{qr}} \leqslant \frac{1}{2}|W_0|$ in all cases.

0. Introduction

0.1. The two-sided cell Ω_{qr}. Let W be a Coxeter group with S its distinguished generator set. In [7], Kazhdan and Lusztig introduced the concept of left, right and two-sided cells in W in order to construct representations of W and the associated Hecke algebra $\mathcal{H}(W)$. In [10], Lusztig further introduced a function $a : W \to \mathbb{N} \cup \{\infty\}$ and proved that if $W = W_a$ is an affine Weyl group, then the function a is constant on any two-sided cell of W_a and $a(z) \leqslant \nu$ for any $z \in W_a$, where ν is half the cardinal of the root system Φ associated to W_a.

Let $W_{(i)} = \{w \in W_a \mid a(w) = i\}$ for any $i \geqslant 0$. It is known that the set $W_{(\nu)}$ forms a single two-sided cell of W_a, which consists of $|W_0|$ left cells (see [19, Theorem 5.2], [20, Theorem 1.1]), where W_0 is the Weyl group of Φ and $|W_0|$ is its cardinal. $W_{(\nu)}$ is actually the lowest one with respect to the partial ordering $\underset{LR}{\leqslant}$ in the set $\mathrm{Cell}(W_a)$ of all two-sided cells of W_a (see 1.2).

Let W_a be an irreducible affine Weyl group (hence the Coxeter graph of W_a is connected). Then by a result of Lusztig in [14, Theorem 4.8], there is a unique two-sided cell (written Ω_{qr}) of W_a which is the second lowest one in $\mathrm{Cell}(W_a)$ with respect to $\underset{LR}{\leqslant}$. The present paper is concerned with Ω_{qr}. First let us propose a conjecture on the number n_{qr} of left cells in Ω_{qr}.

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Conjecture 0.2. *Let W_a be an irreducible affine Weyl group with W_0 the associated Weyl group. Then* $n_{\mathrm{qr}} = \frac{1}{2}|W_0|$.

0.3. Some known cases. Conjecture 0.2 is true when W_a is either $\widetilde{A}_{n-1}$, $n \geqslant 2$, or of rank $\leqslant 4$. In fact, when $W_a = \widetilde{A}_{n-1}$, we see by [16, Theorem 15.1] and [11, Theorem 3.4] that there is a bijection σ from the set $\mathrm{Cell}(\widetilde{A}_{n-1})$ to the set Λ_n of partitions of n. Then $\Omega_{\mathrm{qr}} = \sigma^{-1}(\lambda)$ with $\lambda = (n-1, 1)$. We have $n_{\mathrm{qr}} = n!/2$ by [16, Theorem 14.4.5]. When the rank of W_a is $\leqslant 4$, the result follows from Lusztig [10] for the rank 2 cases, Bédard [2] for $\widetilde{C}_3$, Du [6] for $\widetilde{B}_3$, Zhang [31] for $\widetilde{B}_4$, Chen [5], Shi [23] for $\widetilde{D}_4$, and Shi [24,26,27] for $\widetilde{F}_4$, $\widetilde{C}_4$.

0.4. Main result. The main result of the paper is to prove the inequality $n_{qr} \leqslant |W_0|/2$ (see Theorem 5.10), obtaining an upper bound for the number n_{qr}.

The proof of our result relies heavily on the properties of alcove forms and sign types for the elements of W_a. The following objects and concepts play important roles in the proof: the sets $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$, $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ and $\mathcal{S}(\Phi)^{\mathrm{s}}_{\mathrm{ad,qr}}$ (see 3.1, 3.4 and 5.5), the W_a-action on them (see 3.3 and 3.6), the sign type decomposition for an element of W_a (see 2.12, the latter is worthy to be further studied).

Lusztig conjectured that any left cell L of W_a should be left-connected (see 5.3 and [1]). It is desirable to verify the left-connectedness of $L \subset \Omega_{\mathrm{qr}}$, but this might rely on the solution of Conjecture 0.2.

0.5. Contents. The contents of the paper are organized as follows. We collect some known results on cells of affine Weyl groups W_a in Section 1. Then in Sections 2–4 we are concerned with alcove forms and sign types for all elements of W_a. The results established in these sections are used in Section 5 to prove the main result of this paper.

1. Some known results on cells in affine Weyl groups

1.1. The relation *x*—*y*. Let $W = (W, S)$ be a Coxeter group with S its distinguished generator set. Let $\leqslant$ be the Bruhat–Chevalley order on W. For $w \in W$, we denote by $\ell(w)$ the length of w. Let $A = \mathbb{Z}[u, u^{-1}]$ be the ring of Laurent polynomials in an indeterminate u with integer coefficients. Let $\mathcal{H}(W)$ be the associated Hecke algebra of W, which is a free A-module with an A-basis $\{T_w \mid w \in W\}$, subject to the multiplication rule:

$$\begin{aligned} &T_x T_y = T_{xy}, \quad \text{if } \ell(x) + \ell(y) = \ell(xy); \\ &T_s^2 = \left(u^{-1} - u\right)T_s + T_e \quad \text{for any } s \in S \end{aligned} \tag{1.1.1}$$

where e is the identity element of W. $\mathcal{H}(W)$ has another A-basis $\{C_w \mid w \in W\}$ given by

$$C_w = \sum_{y \leqslant w} u^{\ell(w)-\ell(y)} P_{y,w}\left(u^{-2}\right) T_y \tag{1.1.2}$$

where $P_{y,w} \in \mathbb{Z}[u]$ for any $y, w \in W$ is known as a *Kazhdan–Lusztig polynomial*. Those polynomials satisfy the following properties: $P_{y,w} = 0$ if $y \nleqslant w$; $P_{w,w} = 1$; $\deg P_{y,w} \leqslant (1/2)(\ell(w) - \ell(y) - 1)$ if $y < w$ (see [7]). For any $y, w \in W$ with $\ell(y) \leqslant \ell(w)$, let $\mu(w, y) = \mu(y, w)$ be the coefficient of $u^{(1/2)(\ell(w)-\ell(y)-1)}$ in $P_{y,w}$. We denote y—w if $\mu(y, w) \neq 0$. It is well known that

$$\text{if } x, y \in W \text{ satisfy } y < x \text{ and } \ell(y) = \ell(x) - 1, \text{ then } y\text{—}x. \tag{1.1.3}$$

1.2. Cells. The preorders $\underset{L}{\leqslant}$, $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$ and the associated equivalence relations $\underset{L}{\sim}$, $\underset{R}{\sim}$, $\underset{LR}{\sim}$ on W are defined as in [7]. The equivalence classes of W with respect to $\underset{L}{\sim}$ (respectively $\underset{R}{\sim}$, $\underset{LR}{\sim}$) are called *left cells* (respectively *right cells, two-sided cells*). The preorder $\underset{L}{\leqslant}$ (respectively, $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$) on W induces a partial order on the set of left cells (respectively, right cells, two-sided cells) of W.

1.3. Affine Weyl group. An affine Weyl group W_a is a Coxeter group which can be realized geometrically as follows. Let G be a connected, adjoint reductive algebraic group over $\mathbb{C}$. We fix a maximal torus T of G. Let X be the group of characters $T \to \mathbb{C}$ and let $\Phi \subset X$ be the root system of G with $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ a choice of simple root system. Then $E = X \otimes_{\mathbb{Z}} \mathbb{R}$ is a Euclidean space with inner product $(\,,\,)$ such that the Weyl group (W_0, S_0) of G with respect to T acts naturally on E and preserves its inner product, where S_0 is the set of simple reflections s_i corresponding to the simple roots α_i, $1 \leqslant i \leqslant l$. We denote by Q the group of all translations T_λ $(\lambda \in X)$ on E: T_λ sends x to $x+\lambda$. Then the semidirect product $W_a = W_0 \ltimes Q$ is called an *affine Weyl group*. Let K be the dual of the type of G. Then we define the type of W_a by $\widetilde{K}$. Sometimes we denote W_a by $W_a(\widetilde{K})$ (or even $\widetilde{K}$ in short) to indicate its type $\widetilde{K}$. There is a canonical homomorphism from W_a to $W_0 : w \mapsto \overline{w}$.

Let $-\alpha_0$ be the highest short root in Φ. Set $s_0 = s_{\alpha_0} T_{-\alpha_0}$ with s_{α_0} the reflection corresponding to α_0. Then $S_a = S_0 \cup \{s_0\}$ forms a distinguished generator set of W_a.

1.4. Some known results on cells. Lusztig defined a function $a : W_a \to \mathbb{N}$ which satisfies the following properties.

(1) $a(z) \leqslant \nu := |\Phi|/2$, for any $z \in W_a$ (see [10, Corollary 7.3]).

(2) If $x \underset{LR}{\leqslant} y$ then $a(x) \geqslant a(y)$. In particular, if $x \underset{LR}{\sim} y$ then $a(x) = a(y)$. So we may define $a(\Gamma)$ on a left, right or two-sided cell Γ of W_a by $a(x)$ for any $x \in \Gamma$ (see [10, Theorem 5.4]).

(3) If $a(x) = a(y)$ and $x \underset{L}{\leqslant} y$ (respectively, $x \underset{R}{\leqslant} y$) then $x \underset{L}{\sim} y$ (respectively, $x \underset{R}{\sim} y$) (see [12, Corollary 1.9]).

(4) For any proper subset I of S_a, the subgroup W_I generated by I is finite, let w_I be the longest element of W_I. Then $a(w_I) = \ell(w_I)$. For any $w \in W_I$, it makes no difference for the value $a(w)$ when w is regarded as an element of W_a or of W_I (see [12, Corollary 1.9]).

For any $w \in W_a$, set

$$\mathcal{L}(w) = \{s \in S_a \mid sw < w\} \quad \text{and} \quad \mathcal{R}(w) = \{s \in S_a \mid ws < w\}.$$

(5) If $x \underset{L}{\leqslant} y$ (respectively, $x \underset{R}{\leqslant} y$), then $\mathcal{R}(x) \supseteq \mathcal{R}(y)$ (respectively, $\mathcal{L}(x) \supseteq \mathcal{L}(y)$). In particular, if $x \underset{L}{\sim} y$ (respectively, $x \underset{R}{\sim} y$), then $\mathcal{R}(x) = \mathcal{R}(y)$ (respectively, $\mathcal{L}(x) = \mathcal{L}(y)$) (see [7, Proposition 2.4]).

(6) By the notation $x = y \cdot z$ $(x, y, z \in W_a)$, we mean $x = yz$ and $\ell(x) = \ell(y) + \ell(z)$. In this case, we call x a *left* (respectively, *right*) *extension* of z (respectively, y), and call z (respectively, y) a *left* (respectively, *right*) *retraction* of x.

If $x = y \cdot z$ then $x \underset{L}{\leqslant} z$ and $x \underset{R}{\leqslant} y$. Hence $a(x) \geqslant a(y), a(z)$ by (2). In particular, if $I \in \{\mathcal{R}(x), \mathcal{L}(x)\}$, then $a(x) \geqslant \ell(w_I)$ (see [10, Proposition 2.4]).

(7) Call an element $s \in S_a$ *special*, if the subgroup of W_a generated by $S_a \setminus \{s\}$ is isomorphic to W_0. Note that s_0 is always special. Let $\widetilde{\alpha}$ be the highest root in Φ. Write $\widetilde{\alpha} := \sum_{i=1}^{l} c_i \alpha_i$ with $c_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant l$. It is well known that for any $1 \leqslant i \leqslant l$, the generator s_i is special if and only if $c_i = 1$. It is known that for any two-sided cell $\Omega \neq \{e\}$ ($e \in W_a$ is the identity element) of W_a and any special $s \in S_a$, the set $Y_s = \{w \in \Omega \mid \mathcal{R}(w) = \{s\}\}$ is non-empty and is a single left cell of W_a (see [15, Theorem 1.2]).

(8) If $W_{(i)}$ (see 0.1) contains w_I for some $I \subset S_a$, then $\{w \in W_{(i)} \mid \mathcal{R}(w) = I\}$ forms a single left cell of W_a (a consequence of (3), (4) and (6)).

(9) $W_{(i)}$ is a single two-sided cell of W_a if $i \in \{0, 1, \nu\}$, which can be described as follows: $W_{(0)} = \{e\}$. $W_{(1)}$ consists of all the non-identity elements of W_a each of which has a unique reduced

expression (see [9, Proposition 3.8]). $W_{(\nu)}$ has three equivalent descriptions (see [19, Theorems 1.1, 2.3 and 2.4]):

(i) the set of all elements of W_a each of which has an expression of the form $x \cdot w_J \cdot y$ for some $x, y \in W_a$ and $J = S_a \setminus \{s\}$ with some special $s \in S_a$ (see (7));
(ii) the lowest two-sided cell of W_a with respect to the partial order $\underset{LR}{\leqslant}$;
(iii) the set of all elements w of W_a whose alcove form $A_w = (k(w;\alpha))_{\alpha\in\Phi}$ satisfies $k(w;\alpha) \neq 0$ for any $\alpha \in \Phi$ (see 2.3).

(10) For any $x \in W_a$, let $\Sigma(x)$ be the set of all left cells Γ of W_a satisfying that there exists some element $y \in \Gamma$ with y—x, $\mathcal{R}(y) \nsubseteq \mathcal{R}(x)$ and $a(y) = a(x)$. Then the condition $x \underset{L}{\sim} y$ in W_a holds if and only if $\mathcal{R}(x) = \mathcal{R}(y)$ and $\Sigma(x) = \Sigma(y)$ (see [24, Theorem 2.1], [25, Theorem 1.4] and [26, Section 5]).

(11) If a left cell L and a right cell R are in the same two-sided cell of W_a, then $L \cap R \neq \emptyset$ (see [13, Subsection 3.1 (k), (l)]).

1.5. The set $\mathcal{D}_k$. Lusztig proved in [12, Subsection 1.3 (a)] that $i(z) := \ell(z) - a(z) - 2\delta(z) \geqslant 0$ for any $z \in W_a$, where $\delta(z) = \deg P_{e,z}$. Set

$$\mathcal{D}_k := \{z \in W_a \mid i(z) = k\}. \tag{1.5.1}$$

Then Lusztig proved the following

Proposition 1.6. *(See [12, Proposition 1.4 (a) and Theorem 1.10].)*

(1) *The set $\mathcal{D}_0$ is a finite subset of W_a consisting of involutions* (following Lusztig, we call the elements in $\mathcal{D}_0$ *distinguished involutions* of W_a).
(2) *Each left* (*respectively, right*) *cell of W_a contains exactly one element in $\mathcal{D}_0$.*

1.7. $\mu(x, y)$, $\mathcal{D}_1^{(1)}$ and $c(W_a)$. For $x, y, z \in W_a$, define $h_{x,y,z} \in A$ by $C_xC_y = \sum_z h_{x,y,z}C_z$. Let $\gamma_{x,y,z}$ (respectively, $\delta_{x,y,z}$) denote the coefficients of $u^{a(z)}$ (respectively, $u^{a(z)-1}$) in $h_{x,y,z}$. Springer got a formula for the function $\mu(x, y)$ with $x^{-1} \neq y$ in W_a as follows (see [22, Subsection 1.17] and [30]):

$$\mu(x,y) = \mu(x^{-1}, y^{-1}) = \sum_{d\in\mathcal{D}_0} \delta_{x^{-1},y,d} + \sum_{f\in\mathcal{D}_1} \gamma_{x^{-1},y,f}\pi(f) \tag{1.7.1}$$

where $\pi(f)$ denotes the leading coefficient of the polynomial $P_{e,f}$. Let

$$\mathcal{D}_1^{(1)} := \{f \in \mathcal{D}_1 \mid \mathcal{L}(f) \neq \mathcal{R}(f)\}. \tag{1.7.2}$$

Then $\mathcal{D}_1^{(1)}$ is a finite subset of W_a by [22, Proposition 4.2]. Let

$$c(W_a) = \max\{\ell(d), \ell(f) \mid d \in \mathcal{D}_0,\ f \in \mathcal{D}_1^{(1)}\}. \tag{1.7.3}$$

Then according to (1.7.1), we get by [22, Subsection 1.20 and Proposition 4.2] the following

Lemma 1.8. *Assume that $x, y \in W_a$ satisfy x—y and $a(x) = a(y)$.*

(1) *Either $x \underset{L}{\sim} y$ or $x \underset{R}{\sim} y$ holds.*
(2) *If $\mathcal{L}(x) \times \mathcal{R}(x) \neq \mathcal{L}(y) \times \mathcal{R}(y)$ in addition, then $|\ell(y) - \ell(x)| \leqslant c(W_a)$.*

The result (1) in Lemma 1.8 is due to Springer (see [30]).

Table 1

W_a	$\widetilde{A}_n$	$\widetilde{C}_n$	$\widetilde{B}_n$	$\widetilde{D}_n$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{E}_8$	$\widetilde{F}_4$	$\widetilde{G}_2$
$a(\Omega_{\mathrm{qr}})$	$\frac{1}{2}(n^2-n)$	n^2-2n+2	n^2-n	n^2-3n+3	25	46	91	16	6

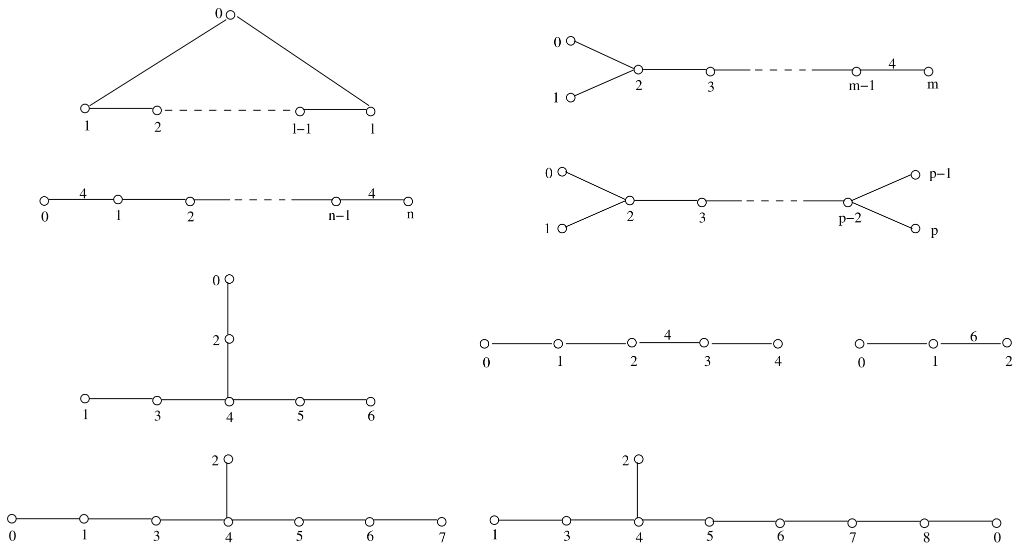

Fig. 1.

1.9. The two-sided cell Ω_{qr}. Let G and W_a be as in 1.3. Then the following result of Lusztig is important to our purpose.

Theorem. *(See [14, Theorem 4.8].) There exists a bijection* $\mathbf{c}: \mathbf{u} \mapsto \mathbf{c}(\mathbf{u})$ *from the set* $\mathfrak{U}$ *of unipotent conjugacy classes in* G *to the set* $\mathrm{Cell}(W_a)$ *of two-sided cells in* W_a*. This bijection satisfies the equation* $a(\mathbf{c}(\mathbf{u})) = \dim \mathfrak{B}_u$*, where* u *is any element in* $\mathbf{u}$*, and* $\dim \mathfrak{B}_u$ *is the dimension of the variety of Borel subgroups of* G *containing* u*.*

Bezrukavnikov recently proved that the bijection $\mathbf{c}$ in the above theorem is order-reversing: a unipotent conjugacy class $\mathbf{u}$ is contained in the closure of a unipotent conjugacy class $\mathbf{v}$ in G if and only if $\mathbf{c}(\mathbf{v}) \underset{\mathrm{LR}}{\leqslant} \mathbf{c}(\mathbf{u})$ (see [3, Theorem 4]). According to the knowledge of unipotent conjugacy classes in algebraic groups (see [4, Chapter 13]), we see by the above theorem that there is a unique lowest two-sided cell $W_{(\nu)}$ (see 0.1) and a unique second lowest two-sided cell (denoted by Ω_{qr}, where the subscript qr of Ω_{qr} is the abbreviation of "quasi-regular" in 3.4) in an irreducible affine Weyl group W_a under the partial order $\underset{\mathrm{LR}}{\leqslant}$. The two-sided cell Ω_{qr} has the form $W_{(i)}$ with i the second largest number in $\{a(\Omega) \mid \Omega \in \mathrm{Cell}(W_a)\}$. More precisely, we have Table 1 for the value $a(\Omega_{\mathrm{qr}})$ by [4, Chapter 13].

Fig. 1 illustrates the Coxeter graphs of types $\widetilde{A}_l$, $\widetilde{B}_m$, $\widetilde{C}_n$, $\widetilde{D}_p$, $\widetilde{E}_6$, $\widetilde{F}_4$, $\widetilde{G}_2$, $\widetilde{E}_7$, $\widetilde{E}_8$ for $l \geqslant 1$, $m > 2$, $n > 1$ and $p \geqslant 4$.

1.10. Define a set

$$\mathcal{S}_X = \begin{cases} \{S_a \setminus \{s_1\}, S_a \setminus \{s_{n-1}\}\}, & \text{if } X = C_n, \\ S_a \setminus \{s_m\}, & \text{if } X = B_m, \\ S_a \setminus \{s_4\}, & \text{if } X = F_4, \\ S_a \setminus \{s_2\}, & \text{if } X = G_2. \end{cases} \tag{1.10.1}$$

By [4, Chapter 13] and Table 1, we see that $\Omega_{\mathrm{qr}} \cap W_0 \neq \emptyset$ if $W_a \in \{\widetilde{A}_l, \widetilde{D}_p, \widetilde{E}_k \mid l \geqslant 1,\ p \geqslant 4,\ k \in \{6,7,8\}\}$ and that $w_J \in \Omega_{\mathrm{qr}}$ for any $J \in \mathcal{S}_X$.

2. Alcoves and sign types

Recall in 1.3 that E is a Euclidean space with inner product $(\,,)$ spanned by an irreducible root system Φ. Let Φ^+ be a choice of positive system in Φ with $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ the simple system of Φ contained in Φ^+. For any $\alpha \in \Phi$, denote by $\alpha^\vee$ the coroot $2\alpha/(\alpha,\alpha)$.

2.1. Alcoves. For $\alpha \in \Phi^+$ and $m \in \mathbb{N}$ and $k \in \mathbb{Z}$, we define a strip of E as follows

$$H^m_{\alpha;k} = H^m_{-\alpha;-k} = \bigl\{v \in E \bigm| k < \bigl(v, \alpha^\vee\bigr) < k+m\bigr\}.$$

Alcoves are connected components of $E - \bigcup_{\alpha\in\Phi^+;\, i\in\mathbb{Z}} H_{\alpha,i}$, where $H_{\alpha,i} := \{v \in E \mid (v,\alpha^\vee) = i\}$ is a hyperplane in E. Any alcove has the form $\bigcap_{\alpha\in\Phi} H^1_{\alpha;k_\alpha}$ with some $k_\alpha \in \mathbb{Z}$.

2.2. Admissible $\boldsymbol{\Phi}$-tuples. An alcove $\bigcap_{\alpha\in\Phi} H^1_{\alpha;k_\alpha}$ of E is determined entirely by a Φ-tuple $(k_\alpha)_{\alpha\in\Phi}$ or a Φ^+-tuple $(k_\alpha)_{\alpha\in\Phi^+}$ over $\mathbb{Z}$. So we can simply write $(k_\alpha)_{\alpha\in\Phi}$ or $(k_\alpha)_{\alpha\in\Phi^+}$ for an alcove $\bigcap_{\alpha\in\Phi} H^1_{\alpha;k_\alpha}$. Note that not any Φ-tuple $(k_\alpha)_{\alpha\in\Phi}$ over $\mathbb{Z}$ gives rise to an alcove of E in the above way. It is so if and only if the following conditions are satisfied.

(a) $k_{-\alpha} = -k_\alpha$ for any $\alpha \in \Phi$;
(b) for any $\alpha, \beta, \gamma \in \Phi$ with $\alpha^\vee + \beta^\vee = \gamma^\vee$, the following inequalities hold (see [29, Theorem 1.1]):

$$k_\alpha + k_\beta \leqslant k_\gamma \leqslant k_\alpha + k_\beta + 1.$$

Let $\mathfrak{A}$ (respectively, $\mathfrak{A}_{\mathrm{ad}}$) be the set of all Φ-tuples $\mathbf{k} := (k_\alpha)_{\alpha\in\Phi}$ over $\mathbb{Z}$ satisfying condition (a) (respectively, conditions (a)–(b)).

Any Φ-tuple $\mathbf{k} := (k_\alpha)_{\alpha\in\Phi}$ in $\mathfrak{A}$ is determined uniquely by the Φ^+-tuple $\mathbf{k}^+ := (k_\alpha)_{\alpha\in\Phi^+}$. So we can identify $\mathbf{k}$ with $\mathbf{k}^+$ and denote both by $\mathbf{k}$.

We have $\mathfrak{A}_{\mathrm{ad}} \subset \mathfrak{A}$. Call an element in $\mathfrak{A}_{\mathrm{ad}}$ an *admissible Φ-tuple*. We identify $\mathfrak{A}_{\mathrm{ad}}$ with the set of all alcoves in E.

2.3. Alcove form of an element in $\boldsymbol{W_a}$. The action of W_a on E induces a permutation on the alcove set $\mathfrak{A}_{\mathrm{ad}}$ which is simply-transitive (see [17, Proposition 5.1]). We know that $A_0 := \bigcap_{\alpha\in\Phi} H^1_{\alpha;0}$ is an alcove in E. Then

$$w \mapsto A_w := (A_0)w$$

is a bijection from W_a to $\mathfrak{A}_{\mathrm{ad}}$. The action of W_a on $\mathfrak{A}_{\mathrm{ad}}$ can be described in terms of admissible Φ-tuples. Hence $A_w := (k(w;\alpha))_{\alpha\in\Phi}$, $w \in W_a$, can be determined uniquely by the following conditions:

(a) $k(e;\alpha) = 0$ for any $\alpha \in \Phi$, where e is the identity of W_a;
(b) $k(s_i; \pm\alpha_i) = \mp 1$ and $k(s_i;\alpha) = 0$ for $0 \leqslant i \leqslant l$ and $\alpha \in \Phi \setminus \{\pm\alpha_i\}$;
(c) Let $w' = ws_i$ for some $0 \leqslant i \leqslant l$. Then $k(w';\alpha) = k(w;(\alpha)\bar{s}_i) + k(s_i;\alpha)$, where $\bar{s}_i = s_i$ if $1 \leqslant i \leqslant l$, and $\bar{s}_0 = s_{\alpha_0}$ (see [17, Proposition 4.2]).

Call both $(k(w;\alpha))_{\alpha\in\Phi}$ and $(k(w;\alpha))_{\alpha\in\Phi^+}$ the *alcove form* of w.

Remark 2.4. (1) The admissibility of a Φ-tuple over $\mathbb{Z}$ is a "local" property on all subsystems of Φ of rank 2. In [17, Theorem 5.2], we defined an admissible Φ-tuple $\mathbf{k} = (k_\alpha)_{\alpha\in\Phi}$ by the requirements

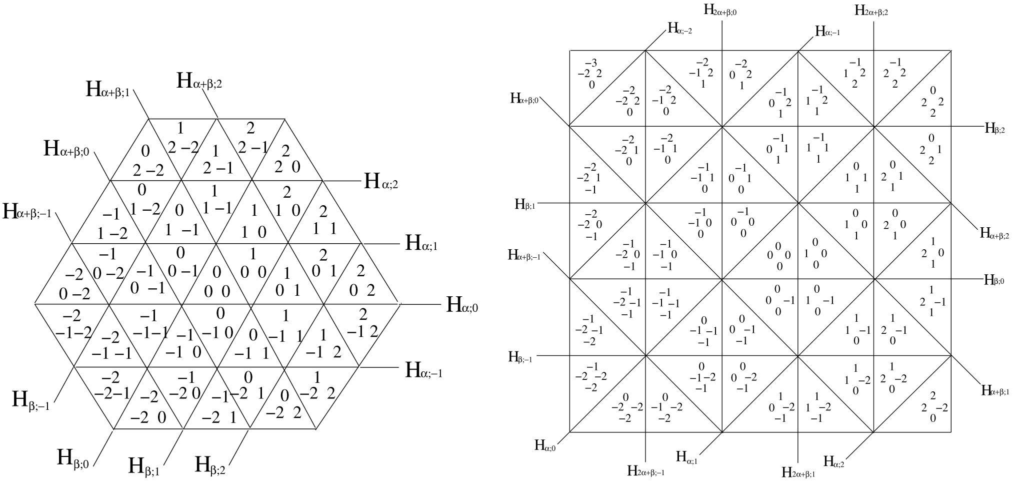

Fig. 2.

$$\begin{aligned}|\alpha|^2k_\alpha+|\beta|^2k_\beta+1&\leqslant|\alpha+\beta|^2(k_{\alpha+\beta}+1)\\&\leqslant|\alpha|^2k_\alpha+|\beta|^2k_\beta+|\alpha|^2+|\beta|^2+|\alpha+\beta|^2-1\end{aligned}\tag{2.4.1}$$

for any $\alpha,\beta\in\Phi$ with $\alpha+\beta\in\Phi$, where $|v|$ denotes the length of a vector v in E. By [29, Theorem 1.1], we see that an admissible Φ-tuple can be re-defined by the condition 2.2 (b) in the place of (2.4.1) for Φ being of classical type (that is, of type A_l, B_m, C_n or D_p for $l>0$ and $m>2$ and $n>1$ and $p>3$) since these two definitions are equivalent. In 2.2, we extend this definition of an admissible Φ-tuple to the case of Φ being an irreducible root system of arbitrary crystallographic type. The present definition of an admissible Φ-tuple is convenient to be used in the subsequent discussion.

(2) By 2.2 (b), **if we set $k_{\alpha^\vee}=k_\alpha$, then it is convenient to replace the root system Φ by its coroot system $\Phi^\vee=\{\alpha^\vee\mid\alpha\in\Phi\}$ as index set when we mention a Φ-tuple**. So from now on, the admissibility condition (b) of a Φ-tuple $\mathbf{k}=(k_\alpha)_{\alpha\in\Phi}$ becomes:

(b′) for any $\alpha,\beta,\gamma\in\Phi$ with $\alpha+\beta=\gamma$, the following inequalities hold:

$$k_\alpha+k_\beta\leqslant k_\gamma\leqslant k_\alpha+k_\beta+1.$$

The root $-\alpha_0$ is now the highest root of Φ instead of the highest short root (see 1.3).

See Fig. 2 for the alcoves $A_w=(k(w;\alpha))_{\alpha\in\Phi^+}$ of E with Φ of types A_2 and B_2, where we write $A_w=\begin{smallmatrix}&k(w;\alpha+\beta)&\\k(w;\alpha)&&k(w;\beta)\end{smallmatrix}$ if Φ has type A_2, and $A_w=\begin{smallmatrix}&k(w;\alpha)&\\k(w;2\alpha+\beta)&&k(w;\beta)\\&k(w;\alpha+\beta)&\end{smallmatrix}$ if Φ has type B_2.

2.5. Operations on $\mathfrak{A}_{\mathbf{ad}}$. Condition 2.3 (c) actually defines a set of operators $\{s_i\mid 0\leqslant i\leqslant l\}$ on the set $\mathfrak{A}_{\mathrm{ad}}$:

$$s_i:(k_\alpha)_{\alpha\in\Phi}\mapsto\left(k_{(\alpha)\bar{s}_i}+k(s_i,\alpha)\right)_{\alpha\in\Phi}.$$

Recall the definition for a left extension of an element $x\in W_a$ in 1.4 (6). The following results on the alcove form $(k(w;\alpha))_{\alpha\in\Phi}$ of $w\in W_a$ are known.

Proposition 2.6. *(See [21, Proposition 4.7].) Let $A_w = (k(w;\alpha))_{\alpha\in\Phi}$ and $A_{w'} = (k(w';\alpha))_{\alpha\in\Phi}$ for $w, w' \in W_a$.*

(a) $\mathcal{R}(w) = \{s_j \in S_a \mid k(w;\alpha_j) < 0\}$.
(b) $\ell(w) = \sum_{\alpha\in\Phi^+} |k(w;\alpha)|$, *where $|x|$ is the absolute value of $x \in \mathbb{Z}$.*
(c) *If $w' = s_j w$ with $0 \leqslant j \leqslant l$, then*

$$k(w';\alpha) = k(w;\alpha) + k\big(s_j;(\alpha)\overline{w}^{-1}\big) \quad \textit{for any } \alpha\in\Phi.$$

(d) *If $w' = ws_j$ with $0 \leqslant j \leqslant l$, then*

$$k(w';\alpha) = k\big(w;(\alpha)s_j\big) + k(s_j;\alpha) \quad \textit{for any } \alpha\in\Phi.$$

(e) *w' is a left extension of w if and only if the inequalities $k(w';\alpha)k(w;\alpha) \geqslant 0$ and $|k(w';\alpha)| \geqslant |k(w;\alpha)|$ hold for any $\alpha\in\Phi$.*

2.7. The set Ξ. Let Ξ be a set consisting of three symbols $+$, $-$, $\bigcirc$. Define a total order $\leqslant$ on Ξ by setting $- < \bigcirc < +$. Also, define a composition "$\cdot$" on Ξ by setting $+\cdot+ = -\cdot- = +$, $+\cdot- = -\cdot+ = -$ and $x\cdot\bigcirc = \bigcirc\cdot x = \bigcirc$ for any $x\in\Xi$.

2.8. Admissible sign types. A Φ-tuple $X = (X_\alpha)_{\alpha\in\Phi}$ over Ξ is called a *Φ-sign type* (or a *sign type* in short) if $X_\alpha \in \Xi$ and $X_{-\alpha} = -X_\alpha$ for any $\alpha\in\Phi$. Let $\mathcal{S}(\Phi)$ be the set of all Φ-sign types. We see by definition that any $(X_\alpha)_{\alpha\in\Phi} \in \mathcal{S}(\Phi)$ is determined uniquely by the Φ^+-sign type $(X_\alpha)_{\alpha\in\Phi^+}$. Hence we may identify $(X_\alpha)_{\alpha\in\Phi}$ with $(X_\alpha)_{\alpha\in\Phi^+}$.

Call $X = (X_\alpha)_{\alpha\in\Phi} \in \mathcal{S}(\Phi)$ *admissible*, if

$$\begin{aligned} -\in\{X_\alpha, X_\beta\} &\implies X_\gamma \leqslant \max\{X_\alpha, X_\beta\}, \\ -\notin\{X_\alpha, X_\beta\} &\implies X_\gamma \geqslant \max\{X_\alpha, X_\beta\} \end{aligned} \tag{2.8.1}$$

for any $\alpha,\beta,\gamma\in\Phi$ with $\gamma = \alpha+\beta$. Let $\mathcal{S}(\Phi)_{\mathrm{ad}}$ be the set of all admissible Φ-sign types.

In geometry, any hyperplane $H_{\alpha;k}$ divides the space E into three pairwise disjoint parts: $H^+_{\alpha;k} = \{v\in E \mid (v,\alpha^\vee) > k\}$ and $H^-_{\alpha;k} = \{v\in E \mid (v,\alpha^\vee) < k\}$ and $H_{\alpha;k}$. Then any admissible Φ-sign type $X = (X_\alpha)_{\alpha\in\Phi}$ can be identified with a connected component of $E - \bigcup_{\alpha\in\Phi^+;\,k\in\{0,1\}} H_{\alpha;k}$, where for any $\alpha\in\Phi^+$, we have $X_\alpha = +$ if $X\subset H^+_{\alpha;1}$; $X_\alpha = -$ if $X \subset H^-_{\alpha;0}$; $X_\alpha = \bigcirc$ if $X\subset H^1_{\alpha;0}$.

Examples 2.9. (1) Write a Φ^+-sign type $X = (X_\alpha)_{\alpha\in\Phi^+}$ in the form $\begin{smallmatrix} & X_{\alpha+\beta} & \\ X_\alpha & & X_\beta \end{smallmatrix}$ if Φ has type A_2, and $\begin{smallmatrix} & X_\beta & \\ X_{\alpha+2\beta} & & X_\alpha \\ & X_{\alpha+\beta} & \end{smallmatrix}$ if Φ has type B_2. See Fig. 3 for the admissible sign types of types A_2, B_2 regarded as the connected components of $E - \bigcup_{\alpha\in\Phi^+;\,k\in\{0,1\}} H_{\alpha;k}$.

(2) When Φ^+ is of type G_2, there are 49 admissible Φ^+-sign types, see [18, Section 2] for the detail.

Remark 2.10. (1) The admissibility of a Φ-sign type is a "local" property on all subsystems of Φ of rank 2. In [18, Section 2], we defined an admissible sign type by displaying all admissible sign types of rank 2. In [28, Subsection 1.5], we re-defined the admissibility of a Φ-sign type by the condition (2.8.1) in the case where Φ is of type A_{n-1}, noting that $\Phi^\vee = \Phi$ in this case. In the present paper, we extend the definition in [28] to the case of Φ being an irreducible root system of ar-

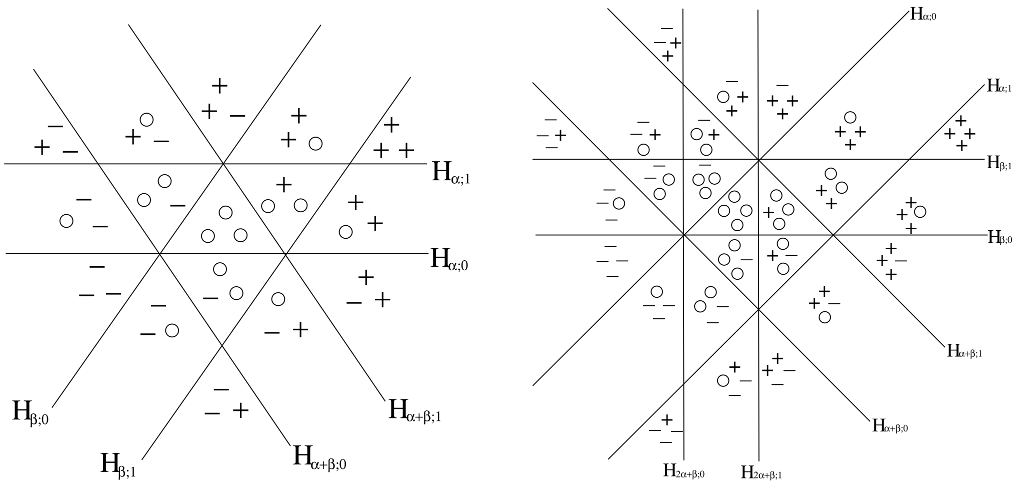

Fig. 3.

bitrary crystallographic type but with the index set $\Phi^\vee$ in the place of Φ. Hence the sign types of type B_2 displayed in Fig. 3 was $\begin{smallmatrix} & X_\beta & \\ X_{\alpha+\beta} & & X_\alpha \\ & X_{2\alpha+\beta} & \end{smallmatrix}$ in [18]. Though the definition given here coincides with that in [18], the present definition is more convenient to be used in the subsequent discussion.

(2) If Φ is an irreducible root system and if $X=(X_\alpha)_{\alpha\in\Phi^+}\in\mathcal{S}(\Phi)_{\mathrm{ad}}$ contains a unique $\bigcirc$-entry (say $X_\beta=\bigcirc$ for some $\beta\in\Phi^+$), then we claim that $X_{(\alpha)s_\beta}=X_\alpha$ for any $\alpha\in\Phi\setminus\{\pm\beta\}$. To prove the claim, we need only to consider the case where $(\alpha)s_\beta\neq\alpha$, then we can reduce ourselves to the case where Φ has rank 2 by (1). But the latter can be checked either by (2.8.1) or by a direct observation on all the related cases in Example 2.9.

2.11. The map ψ. For any $\mathbf{k}=(k_\alpha)_{\alpha\in\Phi^+}\in\mathfrak{A}_{\mathrm{ad}}$, define a Φ^+-sign type $\psi(\mathbf{k})=(X_\alpha)_{\alpha\in\Phi^+}$ by setting

$$X_\alpha=\begin{cases} +, & \text{if } k_\alpha>0,\\ -, & \text{if } k_\alpha<0,\\ \bigcirc, & \text{if } k_\alpha=0.\end{cases}$$

Then we have $\psi(\mathbf{k})\in\mathcal{S}(\Phi)_{\mathrm{ad}}$ by 2.4 (b′) and (2.8.1). This defines a map $\psi:\mathfrak{A}_{\mathrm{ad}}\to\mathcal{S}(\Phi)_{\mathrm{ad}}$, which is surjective (see [18, Theorem 2.1]).

In geometry, for any $X\in\mathcal{S}(\Phi)_{\mathrm{ad}}$, $\psi^{-1}(X)$ is exactly all the alcoves of E contained in the connected component of $E-\bigcup_{\alpha\in\Phi^+;\,k\in\{0,1\}}H_{\alpha;k}$ associated to X.

2.12. Sign type decomposition. By [18, Proposition 7.2], we see that there exists a unique element w in $\psi^{-1}(X)$ for every $X\in\mathcal{S}(\Phi)_{\mathrm{ad}}$ such that any $x\in\psi^{-1}(X)$ is a left extension of w, denote such an element w by w_X.

Any $w\in W_a$ can be written uniquely in the form

$$w=w_{X_1}w_{X_2}\cdots w_{X_r} \tag{2.12.1}$$

for some $X_i\in\mathcal{S}(\Phi)_{\mathrm{ad}}$ with $X_i=\psi(w_{X_1}w_{X_2}\cdots w_{X_i})$ for any $1\leqslant i\leqslant r$. Call (2.12.1) the *sign type decomposition* (or s.t.d. in short) of w.

Corollary 2.13. *Let $w \in W_a$ with s.t.d. $w = w_{X_1} w_{X_2} \cdots w_{X_r}$. Then*

(a) $\ell(w) = \sum_{i=1}^{r} \ell(w_{X_i})$;
(b) $\mathcal{R}(w) = \mathcal{R}(w_{X_r})$;
(c) $\psi(w w_{X_r}^{-1} \cdots w_{X_{j+1}}^{-1}) = X_j$ *for any* $1 < j < r$.

Proof. (a) follows by repeatedly applying Proposition 2.6 (e). Then (b) is a direct consequence of Proposition 2.6 (a). Finally, (c) follows by the definition of s.t.d. □

By Proposition 2.6 (b)–(d), we get the following

Corollary 2.14. *If $w' = wx$ (respectively, $w' = xw$) with $w', w, x \in W_a$ then $k(w;\alpha) - \ell(x) \leqslant k(w';(\alpha)\bar{x}^{-1}) \leqslant k(w;\alpha) + \ell(x)$ (respectively, $k(w;\alpha) - \ell(x) \leqslant k(w';\alpha) \leqslant k(w;\alpha) + \ell(x)$) for any $\alpha \in \Phi$.*

3. The set $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$

3.1. The sets $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$, $\mathcal{S}(\Phi)_{\mathrm{ad,dom}}$ and $\mathcal{S}(\Phi)_{\mathrm{ad,atd}}$. A Φ-sign type $X = (X_\alpha)_{\alpha\in\Phi}$ is called *regular* (respectively, *dominant*, *anti-dominant*) if X_α is in $\{+,-\}$ (respectively, $\{\bigcirc,+\}$, $\{\bigcirc,-\}$) for any $\alpha \in \Phi^+$. Let $\mathcal{S}(\Phi)_{\mathrm{reg}}$ (respectively, $\mathcal{S}(\Phi)_{\mathrm{dom}}$, $\mathcal{S}(\Phi)_{\mathrm{atd}}$) be the set of all regular (respectively, dominant, anti-dominant) Φ-sign types and let $\mathcal{S}(\Phi)_{\mathrm{ad,reg}} = \mathcal{S}(\Phi)_{\mathrm{reg}} \cap \mathcal{S}(\Phi)_{\mathrm{ad}}$, $\mathcal{S}(\Phi)_{\mathrm{ad,dom}} = \mathcal{S}(\Phi)_{\mathrm{dom}} \cap \mathcal{S}(\Phi)_{\mathrm{ad}}$ and $\mathcal{S}(\Phi)_{\mathrm{ad,atd}} = \mathcal{S}(\Phi)_{\mathrm{atd}} \cap \mathcal{S}(\Phi)_{\mathrm{ad}}$. By (2.8.1), we see that $X = (X_\alpha)_{\alpha\in\Phi^+} \in \mathcal{S}(\Phi)_{\mathrm{reg}}$ is in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ if and only if for any $\alpha, \beta, \gamma \in \Phi$ with $\gamma = \alpha + \beta$,

$$\text{the equation } X_\alpha = X_\beta \text{ implies } X_\gamma = X_\alpha. \tag{3.1.1}$$

Actually, the set $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ is in 1–1 correspondence with the set of Weyl chambers in the Euclidean space E spanned by the root system Φ. In particular,

$$\left|\mathcal{S}(\Phi)_{\mathrm{ad,reg}}\right| = |W_0|. \tag{3.1.2}$$

Thus the set $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ is in 1–1 correspondence with the left cells in the lowest two-sided cell $W_{(\nu)}$ (see [20, Corollary 1.2]).

Proposition 3.2. *A regular Φ-sign type $X = (X_\alpha)_{\alpha\in\Phi}$ is in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ if and only if $\{X_\alpha \alpha \mid \alpha \in \Phi^+\}$ forms a positive system of Φ, where $X_\alpha\alpha$ is α if $X_\alpha = +$ and $-\alpha$ if $X_\alpha = -$.*

Proof. Note that a subset F of Φ forms a positive system if and only if the following two conditions hold:

(i) $\{\alpha, -\alpha\} \cap F$ contains exactly one element for any $\alpha \in \Phi$;
(ii) for any $\alpha \neq \beta$ in F, the condition $\alpha + \beta \in \Phi$ implies $\alpha + \beta \in F$.

Then our result follows by (3.1.1). □

3.3. W_a-action on $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$. For any $w \in W_a$ and any $X = (X_\alpha)_{\alpha\in\Phi} \in \mathcal{S}(\Phi)_{\mathrm{ad,reg}}$, we define $Y = (Y_\alpha)_{\alpha\in\Phi} = (X)w$ by setting

$$Y_\alpha = X_{(\alpha)\overline{w}^{-1}}. \tag{3.3.1}$$

We see that if a set F is a positive system of Φ then so is the set $(F)\overline{w}$. Hence $Y \in \mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ by Proposition 3.2. This defines an action of W_a on the set $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$. The action of W_a can be factored through W_0 and is coincident to the action of W_0 on Weyl chambers in E, the latter action is simply-transitive. Hence we can identify W_0 with $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ as sets, where the identity element of W_0 corresponds to the dominant regular sign type, while the longest element w_0 of W_0 corresponds to the anti-dominant regular sign type. In general, if $w \in W_0$ corresponds to $X^w = (X^w_\alpha)_{\alpha\in\Phi}$ then we see by (3.3.1) that $\{\alpha \in \Phi^+ \mid X^w_\alpha = -\}$ is exactly the set of all positive roots which are sent into the negative system Φ^- by w^{-1}.

3.4. The set $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$. A Φ-sign type $X = (X_\alpha)_{\alpha\in\Phi}$ is called *quasi-regular*, if there exists a unique $\gamma \in \Phi^+$ such that $X_\gamma = \bigcirc$ and $X_\beta \neq \bigcirc$ for any $\beta \in \Phi^+ \setminus \{\gamma\}$. In this case, denote γ by $\gamma(X)$. Let $\mathcal{S}(\Phi)_{\mathrm{qr}}$ be the set of all quasi-regular Φ-sign types and let $\mathcal{S}(\Phi)_{\mathrm{ad,qr}} := \mathcal{S}(\Phi)_{\mathrm{qr}} \cap \mathcal{S}(\Phi)_{\mathrm{ad}}$.

For any $X = (X_\alpha)_{\alpha\in\Phi^+} \in \mathcal{S}(\Phi)_{\mathrm{qr}}$, define $X^+ = (X^+_\alpha)_{\alpha\in\Phi^+}$ and $X^- = (X^-_\alpha)_{\alpha\in\Phi^+}$ to be the Φ^+-sign types by setting $X^+_\alpha = X^-_\alpha = X_\alpha$ for any $\alpha \in \Phi^+ \setminus \{\gamma(X)\}$ and $X^+_{\gamma(X)} = +$, $X^-_{\gamma(X)} = -$.

Lemma 3.5. *$X = (X_\alpha)_{\alpha\in\Phi^+} \in \mathcal{S}(\Phi)_{\mathrm{qr}}$ is in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ if and only if both X^+ and X^- are in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$.*

Proof. If $X = (X_\alpha)_{\alpha\in\Phi^+} \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ then we have $\{X_\alpha, X_\beta\} = \{+,-\}$ for any $\alpha, \beta \in \Phi$ with $\alpha+\beta = \gamma(X)$ by (2.8.1). This implies $X^+, X^- \in \mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ again by (2.8.1). Conversely, if $X^+, X^- \in \mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ then $X^+_{\gamma(X)} = +$ and $X^-_{\gamma(X)} = -$. If $\alpha, \beta \in \Phi$ satisfy $\alpha+\beta = \gamma(X)$ then $+ \in \{X^+_\alpha, X^+_\beta\}$ and $- \in \{X^-_\alpha, X^-_\beta\}$ by (2.8.1). This implies $\{X_\alpha, X_\beta\} = \{+,-\}$ by the fact $X_\alpha = X^+_\alpha = X^-_\alpha$ and $X_\beta = X^+_\beta = X^-_\beta$. Hence $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$. $\square$

3.6. W_a-action on $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$. By Lemma 3.5, we can define two maps $\phi_+, \phi_- : \mathcal{S}(\Phi)_{\mathrm{ad,qr}} \to \mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ by setting $\phi_+(X) = X^+$ and $\phi_-(X) = X^-$ for any $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$.

We claim that the union of the images of both ϕ_+ and ϕ_- covers the whole set $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$. For, take any $X \in \mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ and any $s = s_\beta$ with $\beta \in \Pi \cup \{\alpha_0\}$. Then $X = X^w$ for some $w \in W_0$ by 3.3. We have $X^w_{(\beta)w} = -X^{sw}_{(\beta)w}$ and $X^{sw}_\delta = X^w_\delta$ for any $\delta \in \Phi^+ \setminus \{\pm(\beta)w\}$. Hence by Lemma 3.5, there is some $Y \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ such that $\{\phi_+(Y), \phi_-(Y)\} = \{X^w, X^{sw}\}$ (more precisely, we have $\gamma(Y) \in \Phi^+ \cap \{\pm(\beta)w\}$, and $X^w = \phi_+(Y)$ if and only if $X^w_{\gamma(Y)} = +$). This proves our claim.

Consider the set $\mathcal{S}(\Phi)^{(0)}_{\mathrm{ad,reg}}$ of all two-elements subsets $\{Y, Y'\}$ in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$, where there exists some $\gamma \in \Phi^+$ such that $Y'_\alpha = Y_\alpha$ for any $\alpha \in \Phi^+ \setminus \{\gamma\}$ and $Y'_\gamma = -Y_\gamma$. By Lemma 3.5, we see that there exists a bijection $\phi : \mathcal{S}(\Phi)_{\mathrm{ad,qr}} \to \mathcal{S}(\Phi)^{(0)}_{\mathrm{ad,reg}}$ which sends $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ to $\{\phi_+(X), \phi_-(X)\}$. Under this bijection, we can identify X with the set $\{\phi_+(X), \phi_-(X)\}$ and further identify $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ with $\mathcal{S}(\Phi)^{(0)}_{\mathrm{ad,reg}}$.

Recall the action of W_a on $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ defined in 3.3. This induces an action of W_a on $\mathcal{S}(\Phi)^{(0)}_{\mathrm{ad,reg}}$ and hence on $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (so that ϕ is W_a-equivariant). For any $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ and $w \in W_a$, the sign type $Y = (Y_\alpha)_{\alpha\in\Phi} = (X)w$ satisfies $Y_\alpha = X_{(\alpha)\overline{w}^{-1}}$ for any $\alpha \in \Phi$. In particular, if $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ satisfies $X_{\alpha_i} = \bigcirc$ for some $0 \leqslant i \leqslant l$ then $(X)s_i = X$ by Remark 2.10 (2). Unlike that on $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$, the action of W_a on $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ is no longer transitive in general.

Lemma 3.7.

(1) *If $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ is dominant* (*respectively, anti-dominant*), *then $\gamma(X) \in \Pi$.*
(2) *Any W_a-orbit $\mathcal{C}$ in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ contains a unique dominant* (*respectively, anti-dominant*) *sign type.*

Proof. (1) The sign type $X = (X_\alpha)_{\alpha\in\Phi^+} \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ has a unique $\bigcirc$-entry $X_{\gamma(X)}$. Suppose $\gamma(X) \notin \Pi$. Then there exist some $\alpha, \beta \in \Phi^+$ with $\alpha+\beta = \gamma(X)$. If X is dominant (respectively, anti-dominant) then $X_\alpha = X_\beta = +$ (respectively, $X_\alpha = X_\beta = -$) by the assumption of $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$, contradicting (2.8.1). This proves (1).

(2) Let $\mathcal{C}$ be a W_a-orbit in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$. Any $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ can be identified with $\phi(X) \in \mathcal{S}(\Phi)^{(0)}_{\mathrm{ad,reg}}$. By the transitivity of the W_a-action on $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$, we see that $\mathcal{C}$ contains some dominant and also some anti-dominant sign types. Assume that both X and Y are dominant (respectively, anti-dominant)

sign types in $\mathcal{C}$ with $X_{\gamma(X)} = Y_{\gamma(Y)} = \bigcirc$ for some $\gamma(X), \gamma(Y) \in \Phi^+$. We have $\gamma(X), \gamma(Y) \in \Pi$ by (1). Since the W_a-action on $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ factors through W_0, there exists some $w \in W_0$ which sends $\Phi^+ \setminus \{\gamma(X)\}$ to $\Phi^+ \setminus \{\gamma(Y)\}$, and sends $\gamma(X)$ into $\{\pm\gamma(Y)\}$. So $w \in \{e, s_{\gamma(X)}\}$. This implies $\gamma(X) = \gamma(Y)$ in either case, hence $X = Y$. $\square$

4. W_a-orbits on $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$

4.1. Sign type $X^{(i)}$. Let $X \in \mathcal{S}(\Phi)_{\mathrm{ad}}$ satisfying $X_{\alpha_i} \in \{+, \bigcirc\}$ for some $0 \leqslant i \leqslant l$. Define $X^{(i)} = (X^{(i)}_\alpha)_{\alpha\in\Phi}$ by setting

$$X^{(i)}_\alpha = \begin{cases} X_{(\alpha)\bar{s}_i}, & \text{if } \alpha \neq \pm\alpha_i, \\ -, & \text{if } \alpha = \alpha_i, \\ +, & \text{if } \alpha = -\alpha_i. \end{cases}$$

Lemma 4.2. *In the above setup, we have $X^{(i)} \in \mathcal{S}(\Phi)_{\mathrm{ad}}$.*

Proof. Since $X \in \mathcal{S}(\Phi)_{\mathrm{ad}}$, there exists some $w \in W_a$ with $\psi(w) = X$ and $s_i \in S_a \setminus \mathcal{R}(w)$ by Proposition 2.6 (a). Let $w' = ws_i$. Then $\psi(w') = X^{(i)}$ by 2.3 (c) and Proposition 2.6 (a), this implies our result. $\square$

4.3. Increasing and decreasing operations. Call the operation $X \mapsto X^{(i)}$ in Lemma 4.2 *an increasing operation* on X at α_i (or *an increasing operation* on X in short). In this case, we also call the reversing operation $X^{(i)} \mapsto X$ a *decreasing operation* on $X^{(i)}$ at α_i (or a *decreasing operation* on $X^{(i)}$ in short). An increasing operation on X at α_i is applicable if and only if $X_{\alpha_i} \in \{+, \bigcirc\}$. Also, a decreasing operation on X at α_i is applicable if and only if $X_{\alpha_i} = -$. A resulting sign type for an increasing operation on X at α_i, when applicable, is always unique, while that for a decreasing operation on X at α_i, when applicable, need not be unique in general, the latter is unique if and only if there exist some $\alpha, \beta \in \Phi$ satisfying $\alpha + \alpha_i = \beta$ and $(X_\alpha, X_\beta) \in \{(+, -), (\bigcirc, \bigcirc)\}$. More precisely, when (X_α, X_β) is $(+, -)$ (respectively, $(\bigcirc, \bigcirc)$), the resulting sign type $Y^{(i)}$ for a decreasing operation on X at α_i satisfies $Y^{(i)}_{\alpha_i} = +$ (respectively, $Y^{(i)}_{\alpha_i} = \bigcirc$) by (2.8.1). In general, for any $X \in \mathcal{S}(\Phi)_{\mathrm{ad}}$ and any $\alpha_i \in \Pi \cup \{\alpha_0\}$, there exists at most one $X^{(i)} \in \mathcal{S}(\Phi)_{\mathrm{ad}}$ which can be obtained from X by either an increasing operation or a decreasing operation at α_i and which contains the same number of $\bigcirc$-entries as X.

Lemma 4.4. *Let $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, $X \in \mathcal{S}(\Phi)_{\mathrm{ad,reg}}$). Then there exists a sequence of Φ-sign types ξ: $X^{(0)} = X, X^{(1)}, \ldots, X^{(r)} = Y$ in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$) with some $r \geqslant 0$, satisfying that*

(i) *$X^{(i)}$ is obtained from $X^{(i-1)}$ by an increasing operation*;
(ii) *$Y = (Y_\alpha)_{\alpha\in\Phi}$ is a dominant Φ-sign type, that is, $Y_\alpha \in \{+, \bigcirc\}$ for any $\alpha \in \Phi^+$.*

Proof. Our first claim is that there exists a sequence of Φ-sign types ξ': $X^{(0)} = X, X^{(1)}, \ldots, X^{(r)}$ in $\mathcal{S}(\Phi)_{\mathrm{ad}}$ such that $X^{(j)}$ is obtained from $X^{(j-1)}$ by an increasing operation for every $1 \leqslant j \leqslant r$ and that $X^{(r)}$ is a dominant Φ-sign type. For, by the assumption of $X \in \mathcal{S}(\Phi)_{\mathrm{ad}}$, we can find some $w \in W_a$ with $\psi(w) = X$. There exists some $x \in W_a$ such that $\ell(wx) = \ell(w) + \ell(x)$ and $\mathcal{R}(wx) = \{s_0\}$ (see [15, Subsection 2.4]). Let $x = s_1 s_2 \cdots s_r$ be a reduced expression of x with $s_i \in S_a$ and define $x_j = ws_1 s_2 \cdots s_j$ for any $0 \leqslant j \leqslant r$ with the convention that $x_0 = w$. Then the sequence $x_0 = w, x_1, \ldots, x_r = wx$ in W_a satisfies $x_i^{-1} x_{i-1} := s_i \in S_a$ and $\ell(x_i) = \ell(x_{i-1}) + 1$ for every $1 \leqslant i \leqslant r$. The corresponding Φ-sign types $X^{(j)} := \psi(x_j)$ for $0 \leqslant j \leqslant r$ satisfy the required property. The claim is proved.

If $X \in \mathcal{S}(\Phi)_{\mathrm{ad,reg}}$, then the sequence ξ' is clearly in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$. Now assume $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$. Let us choose such a sequence of Φ-sign types with r smallest possible.

Our second claim is that the sequence ξ' is in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$. For otherwise, there exists some $1 \leqslant i \leqslant r$ such that $X^{(j)} \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ and $X^{(l)} \in \mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ for any $0 \leqslant j < i \leqslant l \leqslant r$. Hence $X^{(i-1)}_{\beta_i} = \bigcirc$ and

$X^{(i)}_{\beta_i} = -$ by our construction, where we assume $\overline{s_h} = s_{\beta_h}$ for any $1 \leqslant h \leqslant r$ and some $\beta_h \in \Pi \cup \{\alpha_0\}$. Since $X^{(i-1)} \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$, we see by (2.8.1) and 4.1 that $X^{(i-1)}_\alpha = -X^{(i-1)}_\gamma$ for any $\alpha, \gamma \in \Phi$ with $\alpha + \gamma = \beta_i$. By 4.1 and Remark 2.10 (2), we see that

$$X^{(i)} \text{ is obtained from } X^{(i-1)} \text{ by replacing the entries } X^{(i-1)}_{\pm\beta_i} = \bigcirc \text{ by } X^{(i)}_{\pm\beta_i} = \mp. \qquad (*)$$

Now define the Φ-sign types $Y^{(0)}, Y^{(1)}, \ldots, Y^{(r-1)}$ by setting $Y^{(j)} = X^{(j)}$ for $0 \leqslant j < i$, and let $Y^{(l)}$ be obtained from $Y^{(l-1)}$ by an increasing operation at β_{l+1} for $i \leqslant l < r$. We see by our construction and by $(*)$ that the $Y^{(h)}$'s are well defined and that for any $i \leqslant h < r$, the Φ^+-sign type $Y^{(h)}$ is either equal to $X^{(h+1)}$ or can be obtained from $X^{(h+1)}$ by replacing an entry $X^{(h+1)}_{\gamma_h} \neq \bigcirc$ by $Y^{(h)}_{\gamma_h} = \bigcirc$ for some $\gamma_h \in \Phi^+$. In particular, $Y^{(r-1)}$ is dominant. But this contradicts the minimality assumption on r. The second claim is proved and hence our result follows. $\square$

Lemma 4.5. *For any X in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$), there exists a sequence of Φ-sign types ξ: $X^{(0)} = X, X^{(1)}, \ldots, X^{(r)}$ in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$) with some $r \geqslant 0$ such that $X^{(j)}$ can be obtained from $X^{(j-1)}$ by an increasing operation for every $1 \leqslant j \leqslant r$ and that $X^{(r)}$ is anti-dominant.*

Proof. Let $X = (X_\alpha)_{\alpha\in\Phi}$. Apply induction on the cardinal $m_+(X)$ of the set $\{\alpha \in \Phi^+ \mid X_\alpha = +\}$. If $m_+(X) = 0$ then X itself is anti-dominant and the result is obvious. Now assume $m_+(X) > 0$. Since X is in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$), there must exist some $\gamma \in \Pi$ with $X_\gamma = +$ by (2.8.1). Let X' be the Φ-sign type obtained from X by an increasing operation at γ. Then X' is in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$) with $m_+(X') = m_+(X) - 1$ by Lemma 4.2 and by the fact that X'_α, $\alpha \in \Phi^+ \setminus \{\gamma\}$ is a permutation of X_α, $\alpha \in \Phi^+ \setminus \{\gamma\}$ and that $X'_\gamma = -$. By inductive hypothesis, there exists a sequence of Φ-sign types ξ': $X^{(0)} = X', X^{(1)}, \ldots, X^{(r)}$ in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$) such that $X^{(j)}$ can be obtained from $X^{(j-1)}$ by an increasing operation for every $1 \leqslant j \leqslant r$ and that $X^{(r)}$ is anti-dominant. Then ξ: $X, X^{(0)} = X', X^{(1)}, \ldots, X^{(r)}$ is a required sequence. $\square$

Lemma 4.6. *For any dominant (respectively, anti-dominant) X in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$) and any $N \in \mathbb{N}$, there exists some $w \in \psi^{-1}(X)$ with $A_w = (k(w;\alpha))_{\alpha\in\Phi}$ such that $|k(w;\beta)| > N$ for any $\beta \in \Phi^+$ with $X_\beta \neq \bigcirc$.*

Proof. First assume that $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ is dominant (respectively, anti-dominant). We see by (2.8.1) that there exists a unique $\gamma \in \Pi$ such that $X_\gamma = \bigcirc$. We shall find some $w \in \psi^{-1}(X)$ with $|k(w;\alpha)| > N$ for any $\alpha \in \Phi^+ \setminus \{\gamma\}$ as follows. Let $k_\gamma = 0$ and $k_\alpha = N+1$ (respectively, $k_\alpha = -(N+1)$) for any $\alpha \in \Pi \setminus \{\gamma\}$. Then let $k_\beta = (N+1)\sum_{\alpha\in\Pi\setminus\{\gamma\}} a_\alpha$ (respectively, $k_\beta = -(N+1)\sum_{\alpha\in\Pi\setminus\{\gamma\}} a_\alpha$) for any $\beta \in \Phi^+ \setminus \{\gamma\}$ with $\beta = \sum_{\alpha\in\Pi} a_\alpha \alpha$. Then we have $\mathbf{k} = (k_\alpha)_{\alpha\in\Phi^+} \in \mathfrak{A}_{\mathrm{ad}}$ by 2.4 (b′). So there exists a unique $w \in W_a$ satisfying $A_w = \mathbf{k}$. Clearly, $w \in \psi^{-1}(X)$ satisfies the required property.

Next assume $X \in \mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ dominant (respectively, anti-dominant). We can prove our result in this case in the same way as above except that now $\mathbf{k} = (k_\alpha)_{\alpha\in\Phi^+} \in \psi^{-1}(X)$ is defined by setting $k_\alpha = N+1$ (respectively, $k_\alpha = -(N+1)$) for any $\alpha \in \Pi$, and setting $k_\beta = (N+1)\sum_{\alpha\in\Pi} a_\alpha$ (respectively, $k_\beta = -(N+1)\sum_{\alpha\in\Pi} a_\alpha$) for any $\beta \in \Phi^+$ with $\beta = \sum_{\alpha\in\Pi} a_\alpha \alpha$. $\square$

Lemma 4.7. *For any $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, $X \in \mathcal{S}(\Phi)_{\mathrm{ad,reg}}$), there exists a sequence ξ: $X^{(0)} = X, X^{(1)}, \ldots, X^{(r)}$ in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$) with some $r \geqslant 0$ such that $X^{(j)}$ is obtained from $X^{(j-1)}$ by a decreasing operation for every $1 \leqslant j \leqslant r$ and that $X^{(r)}$ is dominant.*

Proof. Let $m_-(X)$ be the cardinal of the set $\{\alpha \in \Phi^+ \mid X_\alpha = -\}$. Apply induction on $m_-(X) \geqslant 0$. If $m_-(X) = 0$ then the result is obvious. Now assume $m_-(X) > 0$. By (2.8.1), there must exist some $\alpha_i \in \Pi$ with $X_{\alpha_i} = -$. Let $s_i := s_{\alpha_i} \in S_a$. By (2.8.1) and the discussion in 4.3, we see that there exists a unique $Y \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$) obtained from X by a decreasing operation at α_i.

Clearly, $m_-(Y) = m_-(X) - 1$. By inductive hypothesis, there exists a sequence $X^{(0)} = Y, X^{(1)}, \ldots, X^{(r)}$ in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$) with $r \geqslant 0$ such that $X^{(j)}$ is obtained from $X^{(j-1)}$ by a decreasing operation for every $1 \leqslant j \leqslant r$ and that $X^{(r)}$ is dominant. Hence $X, X^{(0)} = Y, X^{(1)}, \ldots, X^{(r)}$ is a required sequence. $\square$

Lemma 4.8. *For any $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, $X \in \mathcal{S}(\Phi)_{\mathrm{ad,reg}}$) and any $N \in \mathbb{N}$, there exists some $w \in \psi^{-1}(X)$ with $A_w = (k(w;\alpha))_{\alpha\in\Phi}$ such that $|k(w;\alpha)| > N$ for any $\alpha \in \Phi$ with $X_\alpha \neq \bigcirc$.*

Proof. By Lemma 4.7 and its proof, we see that there exists a sequence of Φ-sign types ξ: $X^{(0)} = X, X^{(1)}, \ldots, X^{(r)}$ in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$) with $r = m_-(X)$ such that $X^{(j)}$ is obtained from $X^{(j-1)}$ by a decreasing operation on $X^{(j-1)}$ at some $\beta_j \in \Pi$ for every $1 \leqslant j \leqslant r$ and that $X^{(r)}$ is dominant. Let $s_i := s_{\beta_i} \in S_a$ for $1 \leqslant i \leqslant r$. By Lemma 4.6, we see that there exists some $x_r \in \psi^{-1}(X^{(r)})$ with $A_{x_r} = (k(x_r;\alpha))_{\alpha\in\Phi}$ such that $|k(x_r;\alpha)| > N$ for any $\alpha \in \Phi$ with $X^{(r)}_\alpha \neq \bigcirc$. Set $x_j := x_{j+1}s_{j+1}$ for any $0 \leqslant j < r$. Then $\ell(x_j) = \ell(x_{j+1}) + 1$ and $\psi(x_l) = X^{(l)}$ for any $0 \leqslant j < r$ and $0 \leqslant l \leqslant r$. So $w := x_0$ satisfies the required properties by Proposition 2.6 (d). $\square$

Theorem 4.9. *Assume that X, Y are in the same W_a-orbit of $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, of $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$). Then Y can be obtained from X by successively applying increasing (or decreasing) operations.*

Proof. By symmetry, it is enough to show that Y can be obtained from X by successively applying increasing operations. By Lemma 3.7, there exists a unique dominant sign type Z in the W_a-orbit containing X. By Lemma 4.4, Z can be obtained from X by successively applying increasing operations. Then by Lemma 4.7, Y can be obtained from Z by successively applying increasing operations. This implies our result. $\square$

Lemma 4.10.

(1) *For any $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ and any $t \in \mathbb{N}$, there exists an element $w \in \psi^{-1}(X)$ such that $\psi(wy) = (X)y$ for any $y \in W_a$ with $\ell(y) \leqslant t$ and $\psi(wy) \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$.*
(2) *For any $X \in \mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ and any $t \in \mathbb{N}$, there exists an element $w \in \psi^{-1}(X)$ such that $\psi(wy) = (X)y$ for any $y \in W_a$ with $\ell(y) \leqslant t$.*

Proof. By Lemma 4.8, there exists some $w \in \psi^{-1}(X)$ such that $|k(w;\alpha)| > t$ for any $\alpha \in \Phi$ with $X_\alpha \neq \bigcirc$. Then w satisfies the required property by Proposition 2.6 (d) and by the definition of the W_a-actions on $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ and $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ (see 3.3 and 3.6 and 4.3). $\square$

Let $n(W_a) := \max\{\ell(w_Y) \mid Y \in \mathcal{S}(\Phi)_{\mathrm{ad}}\}$.

Proposition 4.11. *Let $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, $X \in \mathcal{S}(\Phi)_{\mathrm{ad,reg}}$). Fix some $t \in \mathbb{N}$.*

(1) *There is some $w \in \psi^{-1}(X)$ with s.t.d.*

$$w = w_{X_1} w_{X_2} \cdots w_{X_r} \quad (\textit{see } 2.12),$$

for some $r \geqslant t$ such that all the sign types X_i, $r+1-t \leqslant i \leqslant r$, are in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$) and belong to the same W_a-orbit.
(2) *If $z \in \psi^{-1}(X)$ has the s.t.d.*

$$z = w_{Z_1} w_{Z_2} \cdots w_{Z_u}$$

with some $u \geqslant t$ and Z_i in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$) for any $u+1-t \leqslant i \leqslant u$. Then $Z_{u+1-i} = X_{r+1-i}$ for all $1 \leqslant i \leqslant t$.

Proof. (1) By Lemma 4.8, there is some $w \in \psi^{-1}(X)$ satisfying $|k(w;\alpha)| \geqslant t \cdot n(W_a)$ for any $\alpha \in \Phi^+$ with $X_\alpha \neq \bigcirc$. We claim that the element w satisfies the required property. By Corollary 2.14, we see that the element $y := ww_X^{-1}$ satisfies $|k(y;\alpha)| \geqslant (t-1) \cdot n(W_a)$ for any $\alpha \in \Phi^+$ with $X_\alpha \neq \bigcirc$. Applying induction on $t \geqslant 1$. We see that the element y has the s.t.d.

$$y = w_{Y_1} w_{Y_2} \cdots w_{Y_{r'}}$$

such that Y_j is in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ (respectively, in $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$) for any $r'+1-(t-1) \leqslant j \leqslant r'$. Now

$$w = w_{Y_1} w_{Y_2} \cdots w_{Y_{r'}} w_X$$

is the s.t.d. of w. So our first assertion follows. Then the second assertion is a consequence of the fact $X_{i-1} = (X_i)w_{X_i}^{-1}$ for $r+2-t \leqslant i \leqslant r$, the latter follows by Lemma 4.10.

(2) Apply induction on $j \geqslant 1$ with $1 \leqslant j \leqslant t$. Since $\psi(z) = \psi(w)$, we have $X_r = Z_u$. Suppose that we have proved that $Z_{u+1-i} = X_{r+1-i}$ for all $1 \leqslant i < j$ with some $1 \leqslant j \leqslant t$. Then $w_{X_{r+2-j}} w_{X_{r+3-j}} \cdots w_{X_r} = w_{Z_{u+2-j}} w_{Z_{u+3-j}} \cdots w_{Z_u}$, denote this common element by y. Then $Z_{u+1-j} = (X)y^{-1} = X_{r+1-j}$ by Lemma 4.10. So our result follows by induction. □

5. The main result

In this section, we shall prove the main result of the paper (i.e., Theorem 5.10). In this theorem, we conclude that the number n_{qr} of left cells of W_a in Ω_{qr} is $\leqslant |W_0|/2$, which gives an upper bound for the number n_{qr}.

Lemma 5.1. *Let $\mathcal{C}$ be a W_a-orbit in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$. If $\psi^{-1}(X) \cap \Omega_{\mathrm{qr}} \neq \emptyset$ for some $X \in \mathcal{C}$, then $\psi^{-1}(Y) \cap \Omega_{\mathrm{qr}} \neq \emptyset$ for all $Y \in \mathcal{C}$.*

Proof. Let $X, Y \in \mathcal{C}$. By Theorem 4.9, it suffices to show that if Y, $Y \neq X$, is obtained from X by an increasing operation at α_i, $0 \leqslant i \leqslant l$, then the condition $\psi^{-1}(X) \cap \Omega_{\mathrm{qr}} \neq \emptyset$ implies $\psi^{-1}(Y) \cap \Omega_{\mathrm{qr}} \neq \emptyset$. Take any $x \in \psi^{-1}(X) \cap \Omega_{\mathrm{qr}}$ and let $y = xs_i$. Then $\ell(y) = \ell(x) + 1$ and $y \in \psi^{-1}(Y)$. By 1.4 (3) and 1.9, we have $y \in \Omega_{\mathrm{qr}}$ since $y \underset{\mathrm{R}}{\leqslant} x$ and $y \notin W_{(\nu)}$. □

Recall the notation $\mathcal{S}(\Phi)^{(0)}_{\mathrm{ad,reg}}$ and the W_a-equivariant bijection $\phi : \mathcal{S}(\Phi)_{\mathrm{ad,qr}} \to \mathcal{S}(\Phi)^{(0)}_{\mathrm{ad,reg}}$ defined in 3.6.

Lemma 5.2. *Each W_a-orbit $\mathcal{C}$ in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ contains exactly $|W_0|/2$ sign types.*

Proof. Identifying each $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ with the two-elements subset $\{\phi_+(X), \phi_-(X)\}$ of $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ in $\mathcal{S}(\Phi)^{(0)}_{\mathrm{ad,reg}}$, we consider the union $U(\mathcal{C})$ of two-elements subsets of $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ in $\mathcal{C}$. We claim that this union is disjoint. For otherwise, there would be some $X \neq Y$ in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ with $Z := \phi_\epsilon(X) = \phi_{\epsilon'}(Y)$ for some $\epsilon, \epsilon' \in \{+, -\}$. We can find some $w \in W_a$ with $(Z)w$ dominant. Since each of the Φ^+-tuples $(X)w$ and $(Y)w$ can be obtained from $(Z)w$ by replacing some entry by $\bigcirc$, we conclude that both $(X)w$ and $(Y)w$ are dominant. So $(X)w = (Y)w$ by Lemma 3.7 and the fact $(X)w, (Y)w \in \mathcal{C}$. This would imply $X = Y$, a contradiction. The claim is proved. Now we have $U(\mathcal{C}) = \mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ by the fact that the action of W_a on $\mathcal{S}(\Phi)_{\mathrm{ad,reg}}$ is transitive. So our result follows by (3.1.2). □

5.3. Left-connected set. A non-empty set $K \subset W_a$ is called *left-connected*, if for any $x, y \in K$, there exists a sequence of elements $x_0 = x, x_1, \ldots, x_r = y$ in K with some $r \geqslant 0$ such that $x_{i-1}x_i^{-1} \in S_a$ for every $1 \leqslant i \leqslant r$.

Lemma 5.4. *For any $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$, the set $\psi^{-1}(X) \cap \Omega_{\mathrm{qr}}$ is either empty or left-connected. So the set $\psi^{-1}(X) \cap \Omega_{\mathrm{qr}}$ is contained in a single left cell whenever it is non-empty.*

Proof. Assume $K := \psi^{-1}(X) \cap \Omega_{\mathrm{qr}} \neq \emptyset$. We must prove that the set K is left-connected. Take any $x, y \in K$. Write $A_x = (k(x;\alpha))_{\alpha\in\Phi^+}$ and $A_y = (k(y;\alpha))_{\alpha\in\Phi^+}$. Let $N = \max\{|k(x;\alpha)|, |k(y;\alpha)| \mid \alpha \in \Phi^+\}$. By Lemma 4.8, we can find some $z \in \psi^{-1}(X)$ with $A_z = (k(z;\alpha))_{\alpha\in\Phi^+}$ such that $|k(z;\alpha)| > N$ for any $\alpha \in \Phi^+$ with $X_\alpha \neq \bigcirc$. Then z is a left extension of both x and y by Proposition 2.6 (e). So there exists a sequence ξ: $x_0 = x, x_1, \ldots, x_r = z, x_{r+1}, \ldots, x_t = y$ in $\psi^{-1}(X)$ with some $t \geqslant r \geqslant 0$ such that $x_{i-1}x_i^{-1} \in S_a$ and $\ell(x_j) = \ell(x_{j-1}) + 1$ and $\ell(x_l) = \ell(x_{l-1}) - 1$ for every $1 \leqslant i \leqslant t$ and $1 \leqslant j \leqslant r < l \leqslant t$. Then $z \underset{L}{\leqslant} x_j \underset{L}{\leqslant} x$ and $z \underset{L}{\leqslant} x_l \underset{L}{\leqslant} y$ and hence $z, x_j \in \Omega_{\mathrm{qr}}$ for every $0 < j < t$ by 1.4 (2), 1.9 and the fact $z \notin W_{(\nu)}$. That is, the sequence ξ lies inside the set K. So K is left-connected. Then the last assertion follows since any left-connected subset in a two-sided cell of W_a is contained in a single left cell by 1.4 (3). $\square$

5.5. The set $\mathcal{S}(\Phi)^{\mathrm{s}}_{\mathrm{ad,qr}}$. Let $\mathcal{S}(\Phi)^{\mathrm{s}}_{\mathrm{ad,qr}}$ be the set of all Φ-sign types X in $\mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ with $X_\gamma = \bigcirc$ for some short root γ if $W_a \in \{\widetilde{B}_m, \widetilde{C}_n, \widetilde{F}_4, \widetilde{G}_2\}$ and let $\mathcal{S}(\Phi)^{\mathrm{s}}_{\mathrm{ad,qr}} = \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ if W_a is of simply-laced type (i.e., $W_a \in \{\widetilde{A}_l, \widetilde{D}_p, \widetilde{E}_m \mid l \geqslant 1;\ p > 3;\ m = 6, 7, 8\}$). Let $\mathcal{S}(\Phi)^{\mathrm{l}}_{\mathrm{ad,qr}} = \mathcal{S}(\Phi)_{\mathrm{ad,qr}} \setminus \mathcal{S}(\Phi)^{\mathrm{s}}_{\mathrm{ad,qr}}$. Note that $\mathcal{S}(\Phi)^{\mathrm{s}}_{\mathrm{ad,qr}}$ (and also $\mathcal{S}(\Phi)^{\mathrm{l}}_{\mathrm{ad,qr}}$ whenever it is non-empty) is stable under the W_a-action. By Lemma 3.7, we see that the number of W_a-orbits in $\mathcal{S}(\Phi)^{\mathrm{s}}_{\mathrm{ad,qr}}$ is 1 if $W_a \in \{\widetilde{B}_m, \widetilde{G}_2\}$, $n-1$ if $W_a = \widetilde{C}_n$, 2 if $W_a = \widetilde{F}_4$, and $|\Pi|$ if W_a is of simply-laced type.

Recall the notation w_X defined in 2.12 and $n(W_a)$ preceding to Proposition 4.11 and $c(W_a)$ in (1.7.3).

Lemma 5.6. *Let $X \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ and $w \in \psi^{-1}(X)$ be with $|k(w;\alpha)| > N$ for any $\alpha \in \Phi$ with $X_\alpha \neq \bigcirc$, where $N \geqslant c(W_a)n(W_a)$. If $x \in W_a$ satisfies x—w, $a(w) = a(x)$, and $\mathcal{L}(x) \times \mathcal{R}(x) \nsubseteq \mathcal{L}(w) \times \mathcal{R}(w)$, then $\psi(x) \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ is in the W_a-orbit containing X.*

Proof. We need only to consider the case where $x < w$, for otherwise we would have either $x = s \cdot w$ or $x = w \cdot s$ for some $s \in S_a$ by [7, Subsections 2.3 (e)–(g)] and the result would be obvious. Since $N \geqslant c(W_a) \cdot n(W_a)$, we see by Proposition 4.11 that w has the s.t.d. below:

$$w = w_{X_1} w_{X_2} \cdots w_{X_r} \tag{5.6.1}$$

where $X_j \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ for any $r - c(W_a) \leqslant j \leqslant r$. Again by Proposition 4.11, we see that all X_j, $r - c(W_a) \leqslant j \leqslant r$, are in the same W_a-orbit. Then as a subexpression of (5.6.1), the element x has a reduced expression which is obtained from (5.6.1) by deleting at most $c(W_a)$ simple reflections by Lemma 1.8 and the assumptions of $a(x) = a(w)$ and $\mathcal{L}(x) \times \mathcal{R}(x) \nsubseteq \mathcal{L}(w) \times \mathcal{R}(w)$. Hence there is at least one (say w_{X_k}) of the w_{X_j}'s, $r - c(W_a) \leqslant j \leqslant r$, in the expression (5.6.1) whose factors are all preserved under this deletion. That is, x has an expression of the form $x = x' \cdot w_{X_k} \cdot y'$ for some $r - c(W_a) \leqslant k \leqslant r$ and some $x', y' \in W_a$ with $\ell(w) - \ell(x) \leqslant c(W_a)$. Let $z = w_{X_{k+1}} w_{X_{k+2}} \cdots w_{X_r}$. Then $\psi(x) = (X_k)y' = (X)z^{-1}y'$ is in the W_a-orbit containing X by Lemma 4.10. $\square$

Proposition 5.7. *$\psi^{-1}(X) \cap \Omega_{\mathrm{qr}} \neq \emptyset$ for any $X \in \mathcal{S}(\Phi)^{\mathrm{s}}_{\mathrm{ad,qr}}$.*

Proof. Assume that $X_\gamma = \bigcirc$ for some short root $\gamma \in \Phi^+$. By Lemma 4.8, we can take, for some $N \in \mathbb{N}$ with $N > m_{\mathrm{qr}} := |\mathcal{S}(\Phi)_{\mathrm{ad,qr}}|$, some $w \in \psi^{-1}(X)$ to satisfy $|k(w;\alpha)| > N$ for any $\alpha \in \Phi^+ \setminus \{\gamma\}$.

(1) First assume that W_a is not of simply-laced type.

When $W_a = \widetilde{C}_n$, we see by Lemma 4.5 that there exists some $y \in W_a$ with $Y := (X)y \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$ anti-dominant. By the assumption of $X_\gamma = \bigcirc$ with γ short, we have $Y_{\alpha_h} = \bigcirc$ and $Y_\alpha = -$ for some short simple root α_h (i.e., $1 \leqslant h < n$) and any $\alpha \in \Phi^+ \setminus \{\alpha_h\}$. Let $Z = (Y)s_{h-1}s_{h-2}\cdots s_1 s_0$. Let $F := \{\alpha_{1i}, \beta_{1j} \mid 1 \leqslant i, j \leqslant n\}$, where $\alpha_{1i} := \alpha_1 + \cdots + \alpha_i$ and $\beta_{1j} := \alpha_1 + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n$ with the convention that $\beta_{1n} := \alpha_{1n}$. Then $Z_{\beta_{1,h+1}} = \bigcirc$; $Z_\alpha = -$ for any $\alpha \in (\Phi^+ \setminus F) \cup \{\beta_{1j} \mid 1 < j \leqslant h\}$; $Z_{\alpha_{1i}} = Z_{\beta_{1j}} = +$ for any $1 \leqslant i \leqslant n$ and for either $h + 2 \leqslant j < n$ or $j = 1$. Hence every $z \in \psi^{-1}(Z)$

satisfies $\mathcal{R}(z) = S_a \setminus \{s_1\}$ and so $\psi^{-1}(Z) \subset \Omega_{\rm qr}$ by 1.10. By Theorem 4.9 and by the assumption on the non-zero entries of A_w, we see that there exists a sequence $x_0 = w, x_1, \ldots, x_r$ in W_a with some $0 \leqslant r < m_{\rm qr}$ such that $x_r \in \psi^{-1}(Z)$ and $x_i^{-1}x_{i-1} \in S_a$ and $\ell(x_i) = \ell(x_{i-1}) - 1$ for every $1 \leqslant i \leqslant r$. This implies $w \in \Omega_{\rm qr}$.

When W_a is $\widetilde{B}_m$ (respectively, $\widetilde{G}_2$), we see by Lemma 3.7 that there exists some $y \in W_a$ with $Y := (X)y \in \mathcal{S}(\Phi)_{\rm ad,qr}$ such that $Y_\alpha = -$ for any α in $(\Pi \setminus \{\alpha_m\}) \cup \{\alpha_0\}$ (respectively, in $\{\alpha_0, \alpha_1\}$). By Theorem 4.9 and by the assumption on the non-zero entries of A_w, we see that there exists a sequence $x_0 = w, x_1, \ldots, x_r$ in W_a with some $0 \leqslant r < m_{\rm qr}$ such that $\psi(x_r) = Y$ and $x_i^{-1}x_{i-1} \in S_a$ and $\ell(x_i) = \ell(x_{i-1}) - 1$ for every $1 \leqslant i \leqslant r$. Clearly, x_r is a left extension of the element w_J with $J = S_a \setminus \{s_m\}$ (respectively, $J = \{s_0, s_1\}$) and hence $x_r \in \Omega_{\rm qr}$ by 1.4 (4) and 1.10. This implies $w \in \Omega_{\rm qr}$ by the facts $w \underset{\rm R}{\leqslant} x_r \underset{\rm L}{\leqslant} w_J$ and $a(w) < \nu$.

When $W_a = \widetilde{F}_4$, we have $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ with α_1, α_2 long and α_3, α_4 short. We see by Lemma 4.5 that there exists some $y \in W_a$ with $Y := (X)y \in \mathcal{S}(\Phi)_{\rm ad,qr}$ anti-dominant. By the assumption of $X_\gamma = \bigcirc$ with γ short, we have $Y_{\alpha_h} = \bigcirc$ and $Y_\alpha = -$ for some $h \in \{3, 4\}$ and for any $\alpha \in \Phi^+ \setminus \{\alpha_h\}$. We can find some $Z \in \mathcal{S}(\Phi)_{\rm ad,qr}$ in the W_a-orbit containing Y such that $Z_{\alpha_i} = -$ for $0 \leqslant i < 4$. More precisely, $Z = (Y)s_0s_1s_2s_3s_2s_1s_0$ if $h = 4$; $Z = (Y)s_0s_1s_2s_3s_2s_1s_0s_4s_3s_2s_1s_0s_3s_2s_1s_3s_2$ if $h = 3$. Hence every $z \in \psi^{-1}(Z)$ satisfies $\mathcal{R}(z) = S_a \setminus \{s_4\}$ and so $\psi^{-1}(Z) \subset \Omega_{\rm qr}$ by 1.10. By the same argument as that in the above paragraph, we conclude that $w \in \Omega_{\rm qr}$.

(2) Next assume that W_a is of simply-laced type, that is, $W_a \in \{\widetilde{A}_l, \widetilde{D}_p, \widetilde{E}_k \mid l \geqslant 1,\ p > 3,\ k = 6, 7, 8\}$. Let E be the set of all non-identity elements $x \in W_0$ each of which has a unique reduced expression. Let w_0 be the longest element in W_0. Then it is well known that the equation $w_0E := \{w_0x \mid x \in E\} = W_0 \cap \Omega_{\rm qr}$ holds by [7, Remark 3.3], [9, Proposition 3.8] and [14, Theorem 4.8] (see also 1.10). Note that the set $\{\psi(x) \mid x \in W_0 \cap \Omega_{\rm qr}\}$ is contained in $\mathcal{S}(\Phi)_{\rm ad,atd}$ and contains the set $\mathcal{S}(\Phi)_{\rm ad,atd} \cap \mathcal{S}(\Phi)_{\rm ad,qr}$. This implies that if $X \in \mathcal{S}(\Phi)_{\rm ad,qr}$ is anti-dominant then any $w \in \psi^{-1}(X)$ is a left extension of some element in w_0E by Proposition 2.6 (e) and hence belongs to $\Omega_{\rm qr}$ by 1.4 (6) and the fact $w \notin W_{(\nu)}$. So by Theorem 4.9, we have $\psi^{-1}(Y) \cap \Omega_{\rm qr} \neq \emptyset$ for any $Y \in \mathcal{S}(\Phi)_{\rm ad,qr}$. $\square$

The assumption "$X \in \mathcal{S}(\Phi)^{\rm s}_{\rm ad,qr}$" is necessary for the validity of Proposition 5.7 (see Proposition 5.9).

5.8. A sequence connecting two left cells in $\Omega_{\rm qr}$. By Proposition 1.6, we see that the number $n_{\rm qr}$ of left cells in $\Omega_{\rm qr}$ is finite. We claim that for any two left cells L, L' in $\Omega_{\rm qr}$, there is a sequence of elements $x_0, x_1, \ldots, x_r$ in $\Omega_{\rm qr}$ with some $r \leqslant n_{\rm qr}$ such that $x_0 \in L$ and $x_r \in L'$ and x_{i-1}—x_i and $\mathcal{R}(x_{i-1}) \nsubseteq \mathcal{R}(x_i)$ for every $1 \leqslant i \leqslant r$. For, take any right cell R in $\Omega_{\rm qr}$. Then $L \cap R \neq \emptyset \neq L' \cap R$ by 1.4 (11). Take any $x \in L \cap R$. Then there is a sequence $x_0 = x, x_1, \ldots, x_r$ in R (hence also in $\Omega_{\rm qr}$) with some $r \geqslant 0$ such that $x_r \in L' \cap R$ and x_{i-1}—x_i and $\mathcal{R}(x_{i-1}) \nsubseteq \mathcal{R}(x_i)$ for every $1 \leqslant i \leqslant r$. By 1.4 (10), we can require $x_i \underset{\rm L}{\nsim} x_j$ for any $i \neq j$ with $0 \leqslant i, j \leqslant r$ in the above sequence. In this case, we have $r \leqslant n_{\rm qr}$. The claim is proved.

Proposition 5.9. *If $w \in W_a(\widetilde{X})$ satisfies $X \in \{B_m, C_n, F_4, G_2 \mid m > 2,\ n > 1\}$ and $\psi(w) \in \mathcal{S}(\Phi)^{\rm l}_{\rm ad,qr}$ then $w \notin \Omega_{\rm qr}$.*

Proof. Recall the notation $\mathcal{S}_X$ in (1.10.1). Notice the following three facts:

(a) The set $\mathcal{S}(\Phi)^{\rm l}_{\rm ad,qr}$ is stable under the W_a-action defined in 3.6;
(b) $w_J \in \Omega_{\rm qr}$ for all $J \in \mathcal{S}_X$ (see 1.10);
(c) For any $w \in W_a(\widetilde{X})$ with $\psi(w) \in \mathcal{S}(\Phi)^{\rm l}_{\rm ad,qr}$, we have $\mathcal{R}(w) \notin \mathcal{S}_X$. This is because any $w \in W_a(\widetilde{X})$ with $\mathcal{R}(w) \in \mathcal{S}_X$ satisfies $k(w; \alpha) \neq 0$ for any long root α in Φ^+.

Suppose that there is some $Y \in \mathcal{S}(\Phi)^{\rm l}_{\rm ad,qr}$ with $\psi^{-1}(Y) \cap \Omega_{\rm qr} \neq \emptyset$. By Lemma 4.8, we can take $x_0 \in \psi^{-1}(Y) \cap \Omega_{\rm qr}$ such that $|k(x_0; \alpha)| > n_{\rm qr} \cdot c(W_a) \cdot n(W_a)$ for any $\alpha \in \Phi^+$ with $Y_\alpha \neq \bigcirc$. There exists a left cell L of W_a in $\Omega_{\rm qr}$ which contains w_J for some $J \in \mathcal{S}_X$ by (b). Then by 5.8 and 1.4 (10), there exist some $x_0, x_1, \ldots, x_r$ in Ω_{qr} with some $r \leqslant n_{\mathrm{qr}}$ and some $x_r \in L$ such that x_{i-1}—x_i and $\mathcal{R}(x_{i-1}) \nsubseteq \mathcal{R}(x_i)$ for every $1 \leqslant i \leqslant r$.

By (a) and Lemma 5.6, we see that all the $\psi(x_i)$, $1 \leqslant i \leqslant r$, are in $\mathcal{S}(\Phi)^{\mathrm{l}}_{\mathrm{ad,qr}}$. But the condition $x_r \in L$ contradicts the fact (c). The result follows. $\square$

Theorem 5.10. *The two-sided cell Ω_{qr} of W_a contains at most $|W_0|/2$ left (respectively, right) cells.*

Proof. Let $\mathcal{C}$ be a W_a-orbit in $\mathcal{S}(\Phi)^{\mathrm{s}}_{\mathrm{ad,qr}}$. Then $\psi^{-1}(X) \cap \Omega_{\mathrm{qr}} \neq \emptyset$ for all $X \in \mathcal{C}$ by Proposition 5.7. By Lemma 5.2, we have $|\mathcal{C}| = |W_0|/2$.

For any $X \in \mathcal{C}$ and $t \in \mathbb{N}$, we can find some $w \in \psi^{-1}(X)$ such that if $x_0 = w, x_1, \ldots, x_t$ is a sequence in W_a satisfying x_{i-1}—x_i, $a(x_{i-1}) = a(x_i)$ and $\mathcal{L}(x_{i-1}) \times \mathcal{R}(x_{i-1}) \nsupseteq \mathcal{L}(x_i) \times \mathcal{R}(x_i)$ for every $1 \leqslant i \leqslant t$ then $\psi(x_j) \in \mathcal{C}$ for all $0 \leqslant j \leqslant t$. Actually, by Lemma 5.6, we can take $w \in \psi^{-1}(X)$ such that

$$\bigl|k(w;\alpha)\bigr| > N := t \cdot c(W_a) \cdot n(W_a) \quad \text{for any } \alpha \in \Phi^+ \text{ with } X_\alpha \neq \bigcirc. \tag{5.10.1}$$

The number n_{qr} of left (respectively, right) cells of W_a in Ω_{qr} is finite by Proposition 1.6. So by the same argument as that in 5.8, we see that for any $w \in \Omega_{\mathrm{qr}}$ and any left (respectively, right) cell L of W_a in Ω_{qr}, there exists a sequence $x_0 = w, x_1, \ldots, x_r$ in Ω_{qr} with some $r \leqslant n_{\mathrm{qr}}$ such that $x_r \in L$ and x_{i-1}—x_i and $\mathcal{L}(x_{i-1}) \times \mathcal{R}(x_{i-1}) \nsupseteq \mathcal{L}(x_i) \times \mathcal{R}(x_i)$ for every $1 \leqslant i \leqslant r$.

Fix $X \in \mathcal{C}$ and take $w \in \psi^{-1}(X)$ to satisfy (5.10.1) with $t = n_{\mathrm{qr}}$. Then we see by 5.8 and Lemma 5.6 that each left (respectively, right) cell L of W_a in Ω_{qr} satisfies $L \cap (\bigcup_{Y \in \mathcal{C}} \psi^{-1}(Y)) \neq \emptyset$. Hence by Lemma 5.4, this further implies that the two-sided cell Ω_{qr} of W_a contains at most $|W_0|/2$ left (respectively, right) cells. $\square$

Remark 5.11. I propose two possible ways to prove that the two-sided cell Ω_{qr} of W_a contains exactly $|W_0|/2$ left cells.

(a) Let D_{qr} be the set of all the distinguished involutions in Ω_{qr}. We need to describe the elements in D_{qr} and then prove the equality $|D_{\mathrm{qr}}| = |W_0|/2$ by Proposition 1.6. In particular, I conjecture that any $d \in D_{\mathrm{qr}}$ has the form $d = x^{-1} \cdot w_J \cdot x$ for some $x \in W_a$ and $J \in \mathcal{S}_X$ with $X \in \{B_m, C_n \mid m > 2,\ n > 1\}$, where $w_J x \in \Omega_{\mathrm{qr}}$ satisfies $s w_J x \notin \Omega_{\mathrm{qr}}$ for any $s \in J$. Just as that in the case of the lowest two-sided cell of W_a (see [19, Theorems 1.1, 2.3 and 2.4], [20, Proposition 4.3]), we need to give a certain geometric interpretation for the elements $w_J x$ (see [8, Proposition 4.2]) and then to enumerate all such kind of elements.

(b) Let $\mathcal{C}$ be a W_a-orbit in $\mathcal{S}(\Phi)^{\mathrm{s}}_{\mathrm{ad,qr}}$. We need to prove that if $X \neq Y$ in $\mathcal{C}$ then $\psi^{-1}(X) \cap \Omega_{\mathrm{qr}}$ and $\psi^{-1}(Y) \cap \Omega_{\mathrm{qr}}$ belong to different left cells. Let $m(Z) = {}^{\#}\{\alpha \in \Phi^+ \mid Z_\alpha = -\}$ for $Z \in \mathcal{S}(\Phi)_{\mathrm{ad,qr}}$. We may assume $m(X) \leqslant m(Y)$ and prove the result by induction on $m(X) \geqslant 0$. We see by 1.4 (7) that the result is true when $m(X) = 0$.

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